Acta Sci. Math., 40 (1978), 237—242

Near normality of a class of transforms

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1. Introduction. The class of bounded linear transformations $T: L^2(\mathbf{R}_+) \to L^2(\mathbf{R}_+)$ which satisfy the functional equation $Tt(a)=t(a)T$, all $a \in \mathbf{R}_+$, where $t(a)$ is the operator $(t(a)f)(x)=f(ax)$, is characterised by

$$T = M^{-1}m[K]M, \quad K \text{ some function in } L^\infty(\mathbf{R}),$$

where M denotes the Mellin transform operator and $m[K]$ denotes multiplication by the function K. Suitably choosing K, another equivalent characterisation, which is the familiar integral representation ([3], [6], [8]) for this class of mappings, is given by

$$\int_0^u Tf(t)\,dt = \int_0^\infty k(ux^{-1})f(x)\,dx, \quad u \in \mathbf{R}_+.$$

Define a mapping S by $S=TR$, where R is the linear mapping $Rf(x)=x^{-1}f(x^{-1})$, $x \in \mathbf{R}_+$, and T is some member of the class of mappings defined above.

Mappings $S=TR$ have been called Watson transforms, and a study of these mappings has been carried out by a large number of authors (see [3], [6], [7], [8], [9], and [10], for further references). In this note we study Watson transforms from the point of view of bounded linear mappings acting on a functional Hilbert space, and show that although the class of Watson transforms is non-normal, it displays a large number of properties enjoyed by normal operators. In particular, we show that the class of Watson transforms consists of centered, normaloid operators for which the concepts of normal, quasi-normal, subnormal, hyponormal, quasi-hyponormal and paranormal coincide. It is shown that reducing subspaces for Watson transforms exist, and that the determination of the spectrum of a member of this class is very much linked with the determination of the spectrum of a normal transformation.

Received November 12, 1975, in revised form July 22, 1976.

2. Preliminaries. Let T, where T satisfies $Tt(a)=t(a)T$, be continuous on $L^2(\mathbf{R}_+)\to L^2(\mathbf{R}_+)$. Then the mapping (Banach space) conjugate to T is given by $T'=RTR$ (see [3]; Kober [3] calls T' the *concrete adjoint* of T). Let J denote the operation of complex conjugation, i.e. $Jg=\bar{g}$. We define the *Hilbert space adjoint,* henceforth called simply the *adjoint*, of T by $T^*=JT'J$. Noting that $P=JR=P^*=P^{-1}$, a simple argument shows that if $S=TR$, then $S^*=JSJ$. Let Q denote the mapping $Qf(t)=f^{\sim}(t)=f(-t), t\in\mathbf{R}$. It is a simple matter to see that (use the following properties of the Mellin transforms: $MR=QM$ and $MJ=JQM$).

Theorem. (cf. [6, Theorem (2.3)]) *If* $S=M^{-1}m[K]MR$, *then* S *is normal* $\Leftrightarrow SS^*=JSS^*J\Leftrightarrow|K^{\sim}|=|K|$. *Furthermore, if* K *is even (i.e.* $K^{\sim}=K$*), then these conditions are equivalent to the implication that* $S=TR=RT$ *for some* T *satisfying the functional equation* $Tt(a)=t(a)T$.

In the sequel we will write S_K for S to denote its dependence on the function K.

3. Main result. Let A be a bounded linear mapping on a Hilbert space H to itself. The mapping A is said to be (i) *normal* if A and A^* commute; (ii) *quasi-normal* if A commutes with A^*A; (iii) *subnormal* if A has a normal extension; (iv) *hyponormal* if $\|A^*f\|\leqq\|Af\|$ for all $f\in H$; (v) *quasi-hyponormal* if $\|A^*Af\|\leqq\|AAf\|$ for all $f\in H$; (vi) *paranormal* if $\|Af\|^2\leqq\|A^2f\|$ for all unit vectors $f\in H$; (vii) *normaloid* if $w(A)=\|A\|$, where $w(A)$ denotes the numerical radius [2, p. 114] of A; (viii) *spectraloid if* $w(A)=r(A)$, where $r(A)$ denotes the spectral radius [2, p. 45] of A. We have the following inclusion relations for these classes of operators:

$$\text{(i)}\subseteqq\text{(ii)}\subseteqq\text{(iii)}\subseteqq\text{(iv)}\subseteqq\text{(v)}\subseteqq\text{(vi)}\subseteqq\text{(vii)}\subseteqq\text{(viii)}.$$

The reverse inclusions, in general, do not hold, and this remains true for Watson transforms. However, a partial result holds for Watson transforms, as we now show.

Theorem 1. S_K *is paranormal if and only if it is normal.*

Proof. Clearly, normality of S_K implies paranormality of S_K. We divide the proof of the reverse implication into three steps.

Step 1: S_K^* *is hyponormal if and only if* $|K^{\sim}|\leqq|K|$. Clearly, S_K^* is hyponormal if and only if $S_KS_K^*-S_K^*S_K\geqq0$; since $S_K^*=M^{-1}m[\overline{K}^{\sim}]MR$, this holds if and only if

$$M^{-1}m[K]MRM^{-1}m[\overline{K}^{\sim}]MR-M^{-1}m[\overline{K}^{\sim}]MRM^{-1}m[K]MR\geqq0,$$

or if and only if $M^{-1}m[|K|^2-|K^{\sim}|^2]M\geqq0$, i.e. if and only if $|K|\geqq|K^{\sim}|$.

Step 2: S_K *is paranormal only if* $|K^{\sim}|\leqq|K|$. It is not very difficult to see (see [5], for example) that a mapping A on a Hilbert space H is paranormal if and only if $A^{*2}A^2+2\lambda A^*A+\lambda^2 I \geqq 0$ for all real λ. Substituting S_K for A, and noting that

$$(S_K^*)^2 = M^{-1}m[\bar{K}^{\sim}\bar{K}]M, \quad S_K^2 = M^{-1}m[KK^{\sim}]M \quad \text{and} \quad S_K^* S_K = M^{-1}m[|K^{\sim}|^2]M,$$

we now see that S_K is paranormal if and only if

$$M^{-1}m[|K|^2|K^{\sim}|^2]M+2\lambda M^{-1}m[|K^{\sim}|^2]M+\lambda^2 I \geqq 0$$

for all real λ, or what is the same (use the definition of a positive operator), $|K|^2|K^{\sim}|^2+2\lambda|K^{\sim}|^2+\lambda^2 \geqq 0$ for all real λ. But the last inequality holds only if $|K^{\sim}| \leqq |K|$.

Step 3: S_K is paranormal only if $|K|=|K^{\sim}|$. From Steps 1 and 2 it now follows that if S_K is paranormal, then S_K^* is hyponormal, and hence that S_K^* is paranormal. Using once again the definition of paranormality we see that S_K^* is paranormal if and only if $|K|^2|K^{\sim}|^2+2\lambda|K|^2+\lambda^2 \geqq 0$ for all real λ. This last inequality clearly implies that $|K| \leqq |K^{\sim}|$, and so we have that $|K|=|K^{\sim}|$.

The proof, once one takes into consideration the fact that S_K is normal if and only if $|K|=|K^{\sim}|$, is now complete.

Remark. The proof of Step 3 can also be deduced from the properties of the self-commutator [2, p. 132] of an operator. By Step 2, if S_K is paranormal then S_K^* is hyponormal, and hence $D=S_K S_K^*-S_K^* S_K=M^{-1}m[|K|^2-|K^{\sim}|^2]M \geqq 0$. Now if $|K|=|K^{\sim}|$, then there is nothing to prove; if, on the other hand, $|K|>|K^{\sim}|$, then the mapping D (clearly a multiplier transform) is invertible. But this is in contradiction with the fact that a positive self-commutator can not be invertible [2, Problem 188].

Since not all Watson transforms are normal (an example of this is provided by the Watson transform S_K for which the function K is given by $K(t)= =2^{it}\Gamma(1/2+\nu/2+it/2)/\Gamma(1/2+\nu/2-it/2)$, $\operatorname{Re}\nu>-1$), and since concepts (i)—(vi) coincide for Watson transforms, the next best thing to happen to the class of Watson transforms (after it has failed to be normal) would be that the members of the class are normaloid. That this indeed is the case is shown by our Theorem 2. The following lemma will be useful (see also [9, p. 24], where the formulae of the lemma are used in the spectral resolution of Watson transforms).

Lemma.

$$(1) \qquad (S_K)^n(S_K^*)^m = \begin{cases} M^{-1}m[K^{n/2}\bar{K}^{m/2}(K^{\sim})^{n/2}(\bar{K}^{\sim})^{m/2}]M, & \text{if } n, m \text{ are positive even integers,} \\ M^{-1}m[K^{(n+1)/2}\bar{K}^{(m+1)/2}(K^{\sim})^{(n-1)/2}(\bar{K}^{\sim})^{(m-1)/2}]M, & \text{if } n, m \text{ are positive odd integers.} \end{cases}$$

Proof. A straightforward calculation, using again the identities $MR=QM$ and $MJ=JQM$, shows that

$$(2)\qquad (S_K)^n = \begin{cases} M^{-1}m[K^{(n+1)/2}(K^{\sim})^{(n-1)/2}]MR, & \text{if } n \text{ is odd,} \\ M^{-1}m[K^{n/2}(K^{\sim})^{n/2}]M, & \text{if } n \text{ is even;} \end{cases}$$

a similar calculation shows that

$$(3)\qquad (S_K^*)^m = \begin{cases} M^{-1}m[(\bar{K}^{\sim})^{(m+1)/2}\bar{K}^{(m-1)/2}]MR, & \text{if } m \text{ is odd,} \\ M^{-1}m[(\bar{K}^{\sim})^{m/2}\bar{K}^{m/2}]M, & \text{if } m \text{ is even.} \end{cases}$$

Substituting in $(S_K)^n(S_K^*)^m$, the lemma follows.

Theorem 2. *S_K is normaloid.*

Proof. To prove the theorem it is enough to show that $\|S_K\|^n=\|(S_K)^n\|$. It is easily seen that $\|S_K\|=\|K\|_\infty$ (cf. [6, Theorem (2.7)]). Since $S_K^*=JS_KJ=$ $=M^{-1}m[\bar{K}^{\sim}]M$, $\|S_K^*\|=\|S_K\|=\|K^{\sim}\|_\infty$. Clearly $\|S_K\|^n=\|K\|_\infty^n$. From (2) of the proof of the preceding lemma

$$\|(S_K)^n\| = \begin{cases} \|K^{(n+1)/2}\|_\infty \|(K^{\sim})^{(n-1)/2}\|_\infty, & \text{if } n \text{ is odd,} \\ \|K^{n/2}\|_\infty \|(K^{\sim})^{n/2}\|_\infty, & \text{if } n \text{ is even,} \end{cases}$$

i.e. $\|(S_K)^n\|=\|K\|_\infty^n$. This completes the proof.

The norm power and power norm equality is satisfied by each hyponormal operator. A further proof of the nice (near normal) behaviour of Watson transforms is provided by the following equality:

$$w(S_KS_G) = w(M^{-1}m[KG^{\sim}]M) = \|K\|_\infty \|G^{\sim}\|_\infty = w(S_K)w(S_G).$$

(In general, $w(AB)\neq w(A)w(B)$ even for normal A and B: the best one can have is $w(AB)\leqq w(A)w(B)$ for normal operators, and $w(AB)\leqq 4w(A)w(B)$ for operators in general [2, p. 116].)

Having seen earlier that S_K and S_K^* do not commute in general, let us see if any commutativity property is satisfied by $(S_K)^n(S_K^*)^n$ and $(S_K^*)^m(S_K)^m$, where n and m are positive integers. We say that a mapping A (on the Hilbert space H) is *binormal* if A^*A and AA^* commute; the mapping A is siad to be *centered* if the operators in the sequence $\dots, A^2(A^*)^2, AA^*, A^*A, (A^*)^2A^2, \dots$ are mutually commuting [4]. (Clearly, a centered operator is in particular binormal.) For Watson transforms we have

Theorem 3. *S_K is a centered operator.*

Proof. Letting $m=n$ in (1), we have that

$$(S_K)^n(S_K^*)^n = \begin{cases} M^{-1}m[|K|^n|K^{\sim}|^n]M, & \text{if } n \text{ is even,} \\ M^{-1}m[|K|^{n+1}|K^{\sim}|^{n-1}]M, & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, by (2) and (3) we have that

$$(S_K^*)^n(S_K)^n=\begin{cases} M^{-1}m[|K|^n|K^{\sim}|^n]M, & \text{if } n \text{ is even,}\\ M^{-1}m[|K|^{n-1}|K^{\sim}|^{n+1}]M, & \text{if } n \text{ is odd.}\end{cases}$$

The mutual commutativity is now obvious.

4. Spectra, reducing subspaces and unitary Watson transforms. The near normality of Watson transforms is manifest in many other properties that they have. Thus, just as for normal transformations, the residual spectrum $\sigma_r(S_K)$ is empty. If A is a normal transform on a functional Hilbert space H, then A can be represented (use the spectral theorem) as a multiplication, induced by a bounded measurable function φ (say), on some L^2 space, and so the spectrum of A $\big(=\sigma(A)\big)$ is the essential range of φ $(=e_r(\varphi))$. The spectral resolution of the class of Watson transforms has been considered by Akutowicz [1] and de Snoo [9]. We have

$$\lambda\in\sigma(S_K) \quad \textit{if and only if} \quad \lambda^2\in\sigma(S_K S_K).$$

This follows from Theorem (3.4) of [6]. Note that $(S_K)^2$ is normal, and that $\sigma(S_K S_K)=e_r(KK^{\sim})$. Another important property that normal transformations have is that there exist, at least one, non-trivial subspaces that reduce the operator. That the same holds for Watson transforms is shown by the following

Theorem 4. *There exists a subspace V of $L^2(\mathbf{R}_+)$ such that V reduces S_K.*

Proof. Let T_G be the mapping $T_G=M^{-1}m[G]M$. Then $T_G S_K=S_K T_G'$. It follows that if G is even, then the linear manifold $L_G=\{g\in L^2(\mathbf{R}_+): g=T_G f$ for some $f\in L^2(\mathbf{R}_+)\}$ is invariant for S_K (see de Snoo [7, Corollaries (2.6) and (2.8)]). The validity of the theorem is now easily deduced upon suitably choosing G so that $V=L_G$ is closed (e.g., let G be the characteristic function of the interval $[-1, 1]$).

Turning now to unitary transforms, it is well known that a normal operator A is unitary if and only if $\sigma(A)$ lies on the unit circle. That a similar result is true for Watson transforms is contained in the following.

Theorem 5. *The following conditions on S_K are equivalent:*
(a) $|K|=1$; (b) $S_K S_K^*=S_K^* S_K=I$; (c) $S_K=JS_K^{-1}J$; (d) $\sigma(S_K)$ *lies on the unit circle.*

Furthermore, if the function k is defined by $xk(x)=M^{-1}\big(K(t)/(1/2-it)\big)(x)$, then these conditions are equivalent to the implication that

$$\text{(e)} \qquad \int_0^\infty k(ax^{-1})\bar{k}(x^{-1})\,dx=\min(a,1), \quad a\in\mathbf{R}_+.$$

(Condition (e) is of course the classical characterisation of unitary Watson transforms (see, for example, [10])).

Proof. That (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) is not difficult to see. Suppose then that (d) is satisfied. Since the mappings S_K are normaloid, $r(S_K)=w(S_K)=\|S_K\|=1$, and so $|K|\leqq 1$. Similarly, $|K^\sim|\leqq 1$. Since $\lambda\in\sigma(S_K)$ if and only if $\lambda^2\in\sigma(S_K^2)$, the normal transformation S_K^2 has spectrum on the unit circle, and so is unitary. But S_K^2 is unitary if and only if $|KK^\sim|=1$. Hence, upon combining with the previous inequalities, $|K|=1$. Thus (d)$\Rightarrow$(a).

To complete the proof, suppose that there is a function k satisfying the hypotheses of the theorem. Then an argument following closely that in [8, p. 56] shows that

$$\int_0^\infty k(tx)\bar{k}(vx)\,dx/x^2 = \min(t, v), \quad t, v\in \mathbf{R}_+,$$

if and only if S_K is unitary. A suitable change of variable now gives (e).

We conclude with the remarks that (i) the condition that $\sigma(S_K)$ lies on the real axis is not, in general, enough to ensure that the Watson transform S_K be self-adjoint; (ii) Watson transforms are, in general, not convexoid. As an example, consider the Hankel transform of order v, $\operatorname{Re} v>-1$.

Finally, I would like to thank the referee for pointing out references [1] and [9], and for making a number of very helpful comments on the original draft of the paper.

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